ASYMPTOTIC THEORY OF STATICS AND DYNAMICS OF ELASTIC CIRCULAR CYLINDRICAL SHELLS

(ASIMPTOTICHESKAIA TEORIIA STATIKI I DINAMIKI UPRUGIKH KRUGOVYKH TSILINDRICHESKIKH OBOLOCHEK)

PMM Vol.26, No.5, 1962, pp. 923-930

U. K. NIGUL
(Tallin)

(Received May 24, 1962)

An investigation of the accuracy of the Kirchhoff-Love theory for statics of arbitrary shells was initiated in [1,2], in which a state of stress was introduced with an exponent p (see Formula (1.8)) not in excess of a limit $p \leqslant a^{-1/2}$ (a is the relative thickness). This work gave rise to the idea of using the Kirchhoff-Love hypothesis to determine the state of stress of a shell with an asymptotic error of the order of a as $a \to 0$. This confirmation appears to be correct for many problems. Nevertheless, this does not mean that all versions of the Kirchhoff-Love theory are equivalent and are firmly grounded in the asymptotic sense for all problems.

It is of interest to note that problems exist in the calculation of circular cylindrical shells for which the simplified versions of the Kirchhoff-Love theory lead to asymptotic errors of the order of a or $a^{1/2}$, and that certain more complicated versions lead to errors even in the principal terms. It is not difficult to show this for such problems if the solutions obtained from different versions of Kirchhoff-Love theory are compared with analytical solutions obtained from the three-dimensional theory of elasticity [3, p.36].

It is shown that for solutions obtained using the Novozhilov theory [4,5], the coefficients of all static terms with an a^2 multiplier are correct, but that for certain other versions in the theory of cylindrical circular shells [6,10] the coefficients of mixed derivatives of the fourth and sixth order are incorrect. It is not difficult to find problems in which these terms play an essential role.

One such problem has been considered by Darevskii [6]. By retaining the Kirchhoff-Love hypothesis as being sufficiently accurate, he constructed a comparatively complex version of the theory for the solution of problems of the above mentioned type. Upon considering a concrete example, Darevskii was led to the conclusion that his theory and that of Novozhilov gave different formulas for the principal displacement terms. In connection with this, Darevskii [6, p.535] came out with the opinion that the elasticity relations in the Novozhilov theory were too oversimplified for the solution of the problem under consideration. It will be shown here that the correct solution in the asymptotic sense will be obtained on the basis of the Novozhilov theory.

The example mentioned shows clearly that problems exist for which a stricter observance of the Kirchhoff-Love hypothesis not only does not lead to more exact solutions, but rather does the opposite. Therefore, it is desirable to formulate a trustworthy foundation on the basis of estimates of accuracy for different versions of the Kirchhoff-Love theory.

A linear theory for the statics and dynamics of circular cylindrical shells is presented in the paper, constructed as an asymptotic approximation to the three-dimensional theory of elasticity as $a \to 0$. It is used to determine a state of stress which varies sufficiently slowly ($a^2p^2 \ll 1$) with an asymptotic error of the order of $\vartheta_0 \sim a^2p^2 + a^2$. We shall call this theory for short the asymptotic theory.

The suggested asymptotic theory represents a further development of results obtained by the author [3] on the basis of power series, applied by Kil'chevskii [11] in a modern form to construct a two-dimensional shell theory, also by Novozhilov and Finkel'shtein [1], and independently applied by Epstein and Kennard [12-16] to construct a dynamic theory for circular cylindrical shells. An asymptotic estimate of errors was introduced in [3], an exact theory for circular cylindrical shells was constructed, and the solutions were investigated.

In contrast to the above work, all unknown quantities are expressed here in terms of a single solution function. This makes the final results more comprehensible and more convenient for application. It is important to emphasize that the asymptotic theory does not depend on any specific properties of the power series method. It permits analysis of various hypotheses and estimates of accuracy for different versions of the Kirchhoff-Love theory. It is shown that the individual hypotheses of the Kirchhoff-Love theory are not basic in the asymptotic sense. Nevertheless, they found application in the Novozhilov theory, they lead to correct final formulas for shear forces and for various other quantities. Mutual compensation of errors is an important positive property of the Novozhilov theory and guarantees correct coefficients in the solutions. Refinement of the relations of elasticity (in the limit the Kirchhoff-Love theory) may damage the effect of mutual compensation of error and lead to more complex but no less exact theory than that, for example, in [6].

At the end of the paper there is provided the result of an analysis of the accuracy of the Novozhilov theory in the static analysis of circular cylindrical shells by trigonometric series.

A class of problems is demonstrated for which the asymptotic error in the Novozhilov theory does not exceed a or $a^{1/2}$. The problem of Darevskii [6] belongs to this class.

1. Basic notation and initial assumptions. We introduce the following notation: E, modulus of elasticity; μ, Poisson's ratio; ρ, density of the material; R_0, radius of the middle surface of the shell; δ, shell thickness; ξ, φ, ζ, nondimensional coordinates corresponding to the length, transverse arc, and radius of the shell; t, the time; τ, a nondimensional time; u_j ($j = 1, 2, 3$), displacements in the ξ, φ, ζ directions, respectively; σ_{jk} (j, $k = 1, 2, 3$), stresses. The relations are:

$$\xi = \frac{x}{R_0}, \qquad \zeta = \frac{R}{R_0}, \qquad \tau = \frac{t}{R_0}\frac{\sqrt{E}}{\sqrt{\rho(1-\mu^2)}}$$

and

$$\nu = \frac{1}{1-\mu}, \qquad b = \frac{\delta}{2R_0}, \qquad a^2 = \frac{1}{3}b^2, \qquad \zeta_* = \zeta - 1$$

The nondimensional displacements u_j^* and stresses σ_{jk}^* are determined from the formulas

$$u_j = s_j R_0 u_j^* \qquad (s_1 = s_3 = 1, \quad s_2 = i = \sqrt{-1}) \tag{1.1}$$

$$\sigma_{jk} = E\,(1-\mu^2)^{-1}\, s_{jk}\sigma_{jk}^* \tag{1.2}$$

$$(s_{jk} = 1 \quad \text{for } jk = 11, 22, 33, 13; \qquad s_{jk} = i \quad \text{for } jk = 12, 23)$$

and the nondimensional forces, moments and transverse forces from the formulas

$$T_{jk}^* = N_{jk}^\circ,\ M_{jk}^* = N_{jk}^1, \qquad Q_j^* = N_{j3}^\circ \qquad (j, k = 1, 2) \tag{1.3}$$

$$N_{jk}^{(n)} = \frac{1}{2b}\int_{-b}^{b} \alpha_j \zeta_*^n \sigma_{jk}^*\, d\zeta \qquad (j = 1, 2;\ k = 1, 2, 3;\ n = 0, 1;\ \alpha_1 = \zeta,\ \alpha_2 = 1) \tag{1.4}$$

We introduce the notation

$$U_{jn} = \frac{1}{2b}\int_{-b}^{b} \zeta\zeta_*^n u_j^*\, d\zeta \qquad (j = 1, 2, 3;\ n = 0, 1, \ldots) \tag{1.5}$$

for "integrated displacements".

We shall consider a shell either unloaded or with a normal load of the form

$$R_0\delta^{-1}(1 \pm b)\,\sigma_{33}^*(\xi, \varphi, \pm b; \tau) = \pm\frac{1}{2}q_* \qquad (1.6)$$

The relation

$$q_* = R_0\delta^{-1}E^{-1}(1-\mu^2)\,q$$

may be substituted for a large class of applications, where q denotes the normal load on the middle surface. (The cases of pure tension and pure torsion are excluded as they have been studied in detail [5].)

The condition is adopted that all states of stress having a sufficiently small index of variation must be constructed with the aid of a single function Φ. It was assumed in [3] that Φ has the form

$$\Phi = \exp(\lambda\xi - \mathrm{i}m\varphi - i\Omega\tau) \qquad (1.7)$$

The coefficients λ, m, Ω, must satisfy the condition

$$1 \gg \vartheta_0 = a^2g^2, \qquad g^2 = p^2 + c_1, \qquad p^2 = |\,c_2\Omega^2 + c_3\lambda^2 + c_4m^2\,| \qquad (1.8)$$
$$c_j = \text{const} \sim 1$$

Condition (1.8) may be considered as defining the concept of "sufficiently small index of variation of a state of stress". It is a basic assumption in all expositions of this paper.

If the equalities

$$\frac{\partial}{\partial\xi}\Phi = \lambda\Phi, \qquad \frac{\partial}{\partial\varphi}\Phi = -\,\mathrm{i}m\Phi, \qquad \frac{\partial}{\partial\tau}\Phi = -\,i\Omega\Phi \qquad (1.9)$$

are recognized as symbolic ways of describing the derivatives, then the Formulas of [3] and of this paper, apply also for a nonexponential function Φ, which are differentiable a sufficient number of times and because of this, they do not increase faster than (1.7). Such functions Φ are often encountered in constructing particular solutions corresponding to surface loads.

2. Solution equation and be basic formulas of the asymptotic theory. The function Φ is determined with an asymptotic error ϑ_0 from the equations [3, p. 43]

$$[d_0 + a^2(d_1 + d_2)]\,\Phi = q_* \qquad (2.1)$$

where

$$d_0 = -\Omega^6 - \tfrac{1}{2}(3-\mu)\,\Omega^4(\lambda^2 - m^2) + \Omega^4 - \tag{2.2}$$
$$- \tfrac{1}{2}(1-\mu)\,\Omega^2[(\lambda^2 - m^2)^2 - (3+2\mu)\lambda^2 + m^2] + \tfrac{1}{2}(1-\mu)(1-\mu^2)\lambda^4$$

$$d_1 = \tfrac{1}{2}(1-\mu)\,[(\lambda^2 - m^2)^4 - 2m^6 + m^4], \qquad d_2 = 2(1-\mu)\lambda^2 m^2(2m^2 - 1)$$

The homogeneous equation (2.1) exhibits the following property. If one pair of the three quantities Ω, λ and m is given such that their absolute magnitudes are essentially less than a^{-1}, then we obtain a characteristic equation whose roots are also essentially less than a^{-1} and, consequently, Φ satisfies the condition (1.8).

Equation (2.1) includes derivatives with respect to ξ and φ up to the eighth order, and with respect to τ up to the sixth order. This is related to the fact that there are only three forms of expansion waves satisfying condition (1.8).

For the middle surface displacements ($\zeta_* = 0$) we get from this system of Equations [3, p.29]

$$u_{j0}^* = v_{j0} + a^2 v_{j1} + \ldots \qquad (j = 1, 2, 3) \tag{2.3}$$

where

$$v_{10} = -\lambda\Big[\mu\Omega^2 + \tfrac{1}{2}\mu(1-\mu)\lambda^2 + \tfrac{1}{2}(1-\mu)m^2\Big]\Phi$$
$$v_{20} = m\Big[\Omega^2 + \tfrac{1}{2}(2+\mu)(1-\mu)\lambda^2 - \tfrac{1}{2}(1-\mu)m^2\Big]\Phi \tag{2.4}$$
$$v_{30} = \Big[\Omega^4 + \tfrac{1}{2}(3-\mu)\,\Omega^2(\lambda^2 - m^2) + \tfrac{1}{2}(1-\mu)(\lambda^2 - m^2)^2\Big]\Phi$$

$$v_{11} = -\lambda\Big[\Omega^2 g^2 - \tfrac{1}{2}(1-\mu^2)\lambda^4 - \tfrac{1}{2}\mu(3-\mu)\lambda^2 m^2 + \tfrac{1}{4}(5-\mu)m^4 +$$
$$+ \tfrac{3}{4}\mu(3-\mu)\lambda^2 + \tfrac{1}{4}(1-2\mu)m^2\Big]\Phi \tag{2.5}$$
$$v_{21} = m\Big[\Omega^2 g^2 - \tfrac{1}{2}(5 - 2\mu - \mu^2)\lambda^4 + \tfrac{1}{4}(13 - 7\mu - 2\mu^2)\lambda^2 m^2 -$$
$$- \tfrac{3}{4}(1-\mu)m^4 + \tfrac{1}{4}(2 + 3\mu + 4\mu^2)\lambda^2 - \tfrac{3}{4}m^2\Big]\Phi$$
$$v_{31} = \Big[\Omega^2(p^4 + p^2) - \tfrac{3}{4}(\lambda^2 - m^2)^3 + \tfrac{3}{4}(2 - 2\mu + \mu^2)\lambda^4 -$$
$$- \tfrac{1}{4}(6 - 10\mu + 5\mu^2)\lambda^2 m^2 + \tfrac{3}{4}(1-\mu)m^4\Big]\Phi$$

On the basis of (2.3) it follows by means of power series [3,pp.46-49] that the field of displacements and strain components $\varepsilon_{kr}{}^*(k,\ r = 1,\ 2)$

is determined with an asymptotic error ϑ_0 by formulas

$$u_j^* = v_{j0} + \zeta_* \psi_j, \qquad \varepsilon_{kr}^* = \varepsilon_{kr} + \zeta_* \varkappa_{kr} \qquad (j = 1, 2, 3; \ k, r = 1, 2) \tag{2.6}$$

where ψ_j, ε_{kr} and κ_{kr} denote, respectively, the angle of rotation, strain components in the middle surface, and the changes in curvature. For these quantities we have the expressions

$$\psi_1 = -\lambda v_{30}, \qquad \psi_2 = v_{20} + m v_{30}, \qquad \psi_3 = -\mu\nu(\varepsilon_{11}{}^{\vee} + \varepsilon_{22}{}^{\vee}) \tag{2.7}$$

$$\varepsilon_{11} = \varepsilon_{11}{}^{\vee}, \qquad \varepsilon_{12} = \varepsilon_{21} = \varepsilon_{12}{}^{\vee} + \tfrac{1}{2}(1+\mu)\lambda m a^2 G$$

$$\varepsilon_{22} = \varepsilon_{22}{}^{\vee} + \tfrac{3}{4}\mu m^2 a^2 G \tag{2.8}$$

$$\varkappa_{11} = \varkappa_{11}{}^{\vee}, \quad \varkappa_{12} = \varkappa_{12}{}^{\vee}, \quad \varkappa_{21} = \varkappa_{12}{}^{\vee} - \varepsilon_{12}{}^{\vee}, \quad \varkappa_{22} = \varkappa_{22}{}^{\vee} + \psi_3 - \varepsilon_{22}{}^{\vee} \tag{2.9}$$

Here

$$\varepsilon_{11}{}^{\vee} = \lambda v_{10}, \qquad \varepsilon_{12}{}^{\vee} = -m v_{10} + \lambda v_{20}, \qquad \varepsilon_{22}{}^{\vee} = m v_{20} + v_{30}$$

$$G = m^2(m^2-1)\Phi, \qquad \varkappa_{11}{}^{\vee} = \lambda\psi_1, \qquad \varkappa_{12}{}^{\vee} = 2\lambda\psi_2, \qquad \varkappa_{22}{}^{\vee} = m\psi_2 \tag{2.10}$$

Calculation of the integrals (1.4) and (1.5) by power series [3, pp. 46-56] and the application of Formulas (2.3) to (2.10) gives the following expressions for the forces, moments, integrated displacements and transverse forces, with an asymptotic error ϑ_0:

$$T_{11}^* = \varepsilon_{11}{}^{\vee} + \mu\varepsilon_{22}{}^{\vee} - \tfrac{1}{2}\mu(1-\mu) m^2 a^2 G$$

$$T_{22}^* = \mu\varepsilon_{11}{}^{\vee} + \varepsilon_{22}{}^{\vee} + (1-\mu)\left(2\lambda^2 - \tfrac{1}{2}m^2\right) a^2 G \tag{2.11}$$

$$T_{12}^* = \tfrac{1}{2}(1-\mu)(\varepsilon_{12}{}^{\vee} + \lambda m a^2 G)$$

$$T_{21}^* = \tfrac{1}{2}(1-\mu)(\varepsilon_{12}{}^{\vee} + \mu\lambda m a^2 G)$$

$$M_{11}^* = a^2(\varkappa_{11} + \mu\varkappa_{22} + \varepsilon_{11}{}^{\vee} + \mu\varepsilon_{22}{}^{\vee} + \mu\nu q_*)$$

$$M_{22}^* = a^2(\mu\varkappa_{11} + \mu\varkappa_{22} + \mu\nu q_*)$$

$$M_{12}^* = \tfrac{1}{2}(1-\mu) a^2 \varkappa_{12}, \qquad M_{21}^* = \tfrac{1}{2}(1-\mu) a^2 \varkappa_{21}$$

$$U_{j0} = v_{j0}, \quad U_{j1} = a^2(v_{j0} + \psi_j) \qquad (j = 1, 2, 3) \tag{2.12}$$

$$Q_1^* = \lambda M_{11}^* + m M_{21}^* + \Omega^2 U_{11}$$
$$Q_2^* = \lambda M_{12}^* - m M_{22}^* + \Omega^2 U_{21} \qquad (2.13)$$

It can be shown that in problems of statics it is expedient to change the form of M_{11}^* and M_{22}^* in Expressions (2.11) into the following:

$$M_{11}^* = a^2 (\varkappa_{11}^{\vee} + \mu\varkappa_{22}^{\vee} + \varepsilon_{11}^{\vee} + \mu\varepsilon_{22}^{\vee})$$
$$M_{22}^* = a^2 (\mu\varkappa_{11}^{\vee} + \varkappa_{22}^{\vee} - \mu\varepsilon_{11}^{\vee} - \varepsilon_{22}^{\vee}) \qquad (2.14)$$

on the basis of (2.1).

Formulas (2.6) to (2.14) express the unknown quantities in term of the solution function Φ; for the sake of brevity they are not written in their expanded form.

For calculation of the coefficients of $a^2 G$ in all expressions, the tangential forces T_{jk}^* (j, k = 1, 2) show zero terms in the power series [3] with multipliers $\zeta_*^{\ 2}$ and in parts of T_{jj}^* (j = 1, 2) except for terms in the stress σ_{33}. Therefore, Expressions (2.6), although sufficiently accurate for determining ε_{kr}^*, do not allow of vindicating Expressions (2.11) in T_{jk}^*.

The role of the normal σ_{33}^* stresses in the calculation of T_{jj}^* and M_{jj}^* may be judged from the following example.

Integration of the relations in the three-dimensional theory of elasticity gives the following:

$$T_{jj}^* = I_{jj}^{\circ} + J_{jj}^{\circ}, \qquad M_{jj}^* = I_{jj}^{1} + J_{jj}^{1} \qquad (j = 1, 2) \qquad (2.15)$$

where

$$I_{jj}^{n} = \frac{1}{2b} \int_{-b}^{b} \zeta_*^{\ n} \alpha_j \{(1-\mu)\,\varepsilon_{jj}(\zeta) + \mu\,[\varepsilon_{11}(\zeta) + \varepsilon_{22}(\zeta)]\}\, d\zeta$$
$$J_{jj}^{n} = \frac{\mu\nu}{2b} \int_{-b}^{b} \zeta_*^{\ n} \alpha_j \sigma_{33}^*\, d\zeta, \qquad \alpha_1 = \zeta, \qquad \alpha_2 = 1 \quad (n = 0, 1) \qquad (2.16)$$

Then on the basis of [3] we have with accuracy up to terms with an a^2 multiplier

$$J_{11}^{\circ} = J_{22}^{\circ} = \nu\mu a^2 \{-(\nu\Omega^2 + \lambda^2 - m^2)(\lambda v_{10} + m v_{20}) - m v_{20} + 2\Omega^2 v_{30}\} =$$

$$= \ldots + \tfrac{1}{2}\mu\,[(2+\mu)\,\lambda^2 - m^2]\,a^2G$$
$$J_{11}{}^1 = J_{22}{}^1 = \nu\mu a^2 q_* \qquad (2.17)$$

Comparison of Expressions (2.11) and (2.15) shows clearly that such a slowly varying state of stress must exist for which the hypothesis $\sigma_{33}{}^* = 0$ is unacceptable in the special form taken.

It is not difficult to show that Expressions (2.11) to (2.13) satisfy the integrated (two-dimensional) equations of equilibrium of the shell within the limits of asymptotic error ϑ_0. The first and second equations are satisfied identically, the third reduces to Equation (2.1), and the fourth and fifth to Formulas (2.13).

Finally, the exposition of asymptotic theory admits of further simplification for the construction of concrete elementary states of stress. Nevertheless, in this paper, only the difference between asymptotic theory and Kirchhoff-Love theory is taken up, since the problem of further simplification is considered in many papers [4,5,7,20].

3. On the asymptotic error in the Kirchhoff-Love theory. Not a single verison of the Kirchhoff-Love theory coincides completely with the asymptotic theory.

Nevertheless, each version is equivalent to the asymptotic theory in those problems for which, to an error ϑ_0, expressions may be discarded which distinguish them from the asymptotic theory.

All consistent versions of the Kirchhoff-Love theory are equivalent to the asymptotic theory for a large class of problems for which it is possible, to an error ϑ_0, to omit the expression for d_2 in Equation (2.1); the expression for a^2G in Formulas (2.8) and (2.11); and the strain components $\varepsilon_{jk}{}^{\nu}$ in Formulas (2.14).

The application of the version of Kirchhoff-Love theory used by Novozhilov [4] justified itself in the asymptotic sense not only for the problem worked but also for many others.

The Novozhilov theory [4,5] coincides up to an error ϑ_0 with the asymptotic theory in parts of the basic equation [3,20] and in parts of the formulas for the quantities v_{j0} ($j = 1, 2, 3$), ψ_k ($k = 1, 2$), ε_{11}, κ_{11}, κ_{12}, κ_{22}, $T_{jk}{}^*$ ($j, k = 1, 2$), $M_{12}{}^*$ (in problems of statics using the simplified basic equation [5, p.230] the Novozhilov theory coincides completely with Equation 2.1)); his theory differs in the formulas for v_{j1} ($j = 1, 2, 3$), ε_{12}, ε_{22}, κ_{21}, $M_{21}{}^*$, $M_{jj}{}^*$, $Q_j{}^*$ ($j = 1, 2$), and besides, it does not determine ψ_3.

We note that ε_{12} does not coincide with the quantity ω in [4,5]. Nevertheless the formula

$$\varepsilon_{12} = (1 - \mu)^{-1} (T_{12}^* + T_{21}^*)$$

follows from (2.8) and (2.11) and permits correct determination of the value of ε_{12} in terms of T_{12}^*, T_{21}^* in the Novozhilov theory.

Other versions of the Kirchhoff-Love theory [6-8] differ from the asymptotic theory more acutely in regard to coefficients in the expression for d_2 in Equation (2.1) and in front of $a^2 G$ in Formulas (2.11) for T_{jk}^*. The success of the Novozhilov theory arises from mutual compensation of errors in the geometry and in simplified elasticity relations. Therefore it is not difficult to demonstrate problems for which the Novozhilov theory is well grounded in the asymptotic sense, and for which other versions of the Kirchhoff-Love theory lead to asymptotic errors of the order of a^0. The author has no knowledge of cases to the contrary.

Analysis of the accuracy of the Novozhilov theory for symmetrical problems (relative to the circle $\xi = 0$) of statics ($\Omega = 0$), for which

$$a^2 m^8 \lesssim 1 \qquad (m = 2, 3, \ldots) \tag{3.1}$$

$$q_* = r(\xi)\, e^{-im\varphi}, \qquad |r| \geqslant \left| \frac{d^n r}{d\xi^n} \right| \qquad (n = 1, 2, \ldots, \infty) \tag{3.2}$$

and in which ε_{12} did not enter into the conditions given on the boundaries $\xi = \pm \xi_0 \sim 1$, gave the following results.

1. If T_{12}^* does not enter the boundary conditions, then the displacements and forces of the basic state of stress are determined with an asymptotic error a.

2. If T_{12}^* and M_{11}^* or T_{12}^* and Q_1^* enter the boundary conditions, then initial data must be at hand by which the displacements and forces in the basic state of stress are determined to an asymptotic error of $a^{1/2}$.

3. Boundary effects for any unknown quantity and the basic state of stress for moments and transverse forces may be shown to be determinable with asymptotic error greater than $a^{1/2}$ only in the case where corresponding displacements or stresses are a or more times the smallest of the predominant displacement or stress.

There is a series of problems in this class for which other versions of the Kirchhoff-Love theory determine displacements and forces for the basic state of stress to an asymptotic error of the order of a^0. One such case was the problem investigated by Darevskii [6].

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Translated by E.Z.S.